



Relative Degree of Nonlinear Counterclockwise Input-Output Systems and the Feedback Counterclockwise Input-Output Property

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SUMMARY: *This article delves into the relative order and feedback characteristics of nonlinear counterclockwise input-output (CCW I-O) systems. Firstly, based on the theory of dissipation and the Lie algebra method, this paper provides sufficient conditions for the nonlinear CCW I-O system to have relative orders 2 at the origin, and proves that the strict CCW I-O system has the minimum phase characteristic. Secondly, this article delves into the necessary and sufficient conditions for nonlinear systems to possess feedback CCW I-O characteristics, which require the system to have a relative order 2 at the origin and be a weakly minimum phase system. These achievements not only expand the application scope of CCW I-O theory, especially for systems with relative order 2, but also provide effective analysis tools and design methods for systems with such characteristics in engineering fields such as flexible structures and nano-positioning. With the help of representative typical examples, the accuracy and practical application value of the theoretical results obtained are comprehensively verified. Future research will focus on relaxing matching conditions and exploring output feedback strategies.*

KEYWORDS: *Counterclockwise input-output property; relative degree; dissipativity; feedback stability*

1 Introduction

The control theory of dissipative systems provides a fundamental theoretical framework for the analysis and synthesis of systems (see [1-4]). Given the system Σ , if there exists a continuously differentiable storage function $V(x) \geq 0$ that satisfies inequality $V(x(t)) \leq V(x(0)) + \int_0^t \omega(t) dt$, then system Σ is said to possess dissipative properties, or to be a dissipative system, where $V(x)$ is referred to as the storage function of system Σ , and $\omega(t)$ is termed the supply rate. Prominent dissipative properties include passivity, the L_2 finite-gain property, and the counterclockwise input-output (CCW I-O) property. The dissipative property can be linked to the state space representation of the system through the storage function. Based on this connection, scholars can consider many control problems, such as system stability, robust stability, and optimization of certain indicators. The dissipative property finds applications in many scientific fields, such as electricity, mechanics, and thermodynamics.

Passivity, a special case of dissipativity where $\omega(t) = u(t)^T y(t)$, provides a powerful foundation for stabilizing negative feedback interconnections of systems with relative degrees

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of 0 and 1 [5]. Stabilization via passivity can be approached by designing a passive controller or by rendering the closed-loop system passive and stable through feedback. For instance, Haddad and Xie demonstrate the design of passive controllers for robust stabilization of uncertain linear systems [6, 7]. Ortega extensively studies control of Euler-Lagrange systems [2], while Byrnes, Hill, and Kokotovic investigate stability of nonlinear systems [8-10]. Reference [11] adopts the strategy of static output feedback, while reference [12] applies the passivation method of fractional order systems.

The theory of passivity can study the stability problem of negative feedback interconnection for systems with a relative order of at most 1. However, passivity-based methods are inapplicable to systems with a relative degree of 2. To address this existing gap, the study introduced the concept of negative imaginary defined for linear time invariant (LTI) systems by Lanzon and Petersen in 2008 [13], equivalent to the CCW I-O property proposed by Angeli in 2006 from a frequency-domain perspective [14]. This feature makes it possible to conduct stability analysis on feedback interconnections of systems with a relative degree of 2. Afterwards, the relevant theories were further expanded, and their application scope extended to systems with poles on the imaginary axis (references [15, 16]), non rationally symmetric systems [17-19], non-proper systems [20], and discrete-time systems [21]. Recent research has made new breakthroughs, extending the CCW I-O concept to nonlinear time-invariant systems and linear time-varying systems (references [22, 23]), which has greatly promoted the development of stability analysis of interconnected systems. Applications of CCW I-O theory are found in flexible structures [24], nano-positioning [25, 26], multi-agent systems [27-29], vehicle platoons [30], non-square systems with polytopic uncertainties [31], robust coordination control of networked vehicles [32] and so on.

The CCW I-O property, as the nonlinear counterpart of the negative-imaginary property [14], offers a significant advantage: it accommodates systems with a relative degree of 2, whereas passivity is restricted to relative degrees of 0 or 1. If a system can be transformed into a CCW I-O system under certain appropriate conditions, then the CCW I-O theory can be utilized to analyze and synthesize it, especially for systems with a relative degree of 2. This capability is crucial for analyzing systems where passivity theory fails. The one central question is: under what circumstances can a system possess the CCW I-O property? The other is: when is the relative degree of a CCW I-O system 2?

This article focus on the feedback CCW I-O properties and relative order of nonlinear systems based on dissipation theory, and using the CCW I-O properties to investigate control problems of the system, which can further help us understand the CCW I-O properties.

2 Nonlinear counterclockwise input-output system

2.1 System Description

Consider the causal nonlinear system:

$$\Sigma: \begin{cases} \dot{x} = g(x)u + f(x) \\ y = h(x) \\ x(0) = x_0, \end{cases} \quad (1)$$

where $u \in \mathcal{U} \subset L_2^n$, $y \in \mathcal{Y} \subset L_2^n$, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \in \mathcal{X} \subset \mathbb{R}^n$, $g: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ and $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$. For the dynamic system Σ represented by (1) defined on the state space \mathcal{X} , \mathcal{U} and \mathcal{Y} are the input and output spaces, which contain continuous bounded functions defined on the semi-

infinite interval $[0, \infty)$.

Below is the definition of CCW I-O property.

Definition 1 (1)[22]: (CCW I-O) The system (1) is CCW I-O, if there exists a continuously differentiable storage function $V(x) \geq 0$ ($V(0) = 0$) satisfying $V(x(t)) \leq u(t)^T \dot{y}(t), \forall t \geq 0$.

(2)[22] **(Input strictly CCW I-O)** The system is input strictly CCW I-O, if there exists a continuously differentiable storage function $V(x) \geq 0$ ($V(0) = 0$) and a scalar $\epsilon > 0$, satisfying $V(x(t)) \leq u(t)^T \dot{y}(t) - \epsilon u(t)^T u(t), \forall t \geq 0$.

(3)[33] **(Output strictly CCW I-O)** The system is output strictly CCW I-O, if there exists a continuously differentiable storage function $V(x) \geq 0$ ($V(0) = 0$) and a scalar $\delta > 0$, satisfying $V(x(t)) \leq u(t)^T \dot{y}(t) - \delta \dot{y}(t)^T \dot{y}(t), \forall t \geq 0$.

2.2 Example of CCW I-O System

A typical structure of a system amenable to CCW I-O analysis is illustrated in Figure 1. Figure 1 shows the standard lightly damped flexible structure, which takes force as input and displacement as output, can usually be modeled as a second-order transfer function sum (possibly infinite sum) form, that is

$$p_{\Delta}(s) = \sum_{i=1}^H \frac{k_i \omega_{n,i}^2}{s^2 + 2\xi_i \omega_{n,i} s + \omega_{n,i}^2},$$

where k_i, ξ_i and $\omega_{n,i}$ are all greater than 0. In practical operations, for ease of control analysis, the aforementioned sum is often decomposed into a summing form that is easier to handle, consisting of a finite sum $p(s)$ and the remaining part sum $\Delta(s)$. Then, the control problem for such a lightly damped flexible structure can be stated as follows: design a controller $C(s)$ to ensure closed-loop stability of $C(s)$ and $p_{\Delta}(s)$, as illustrated in Figure 1 below:

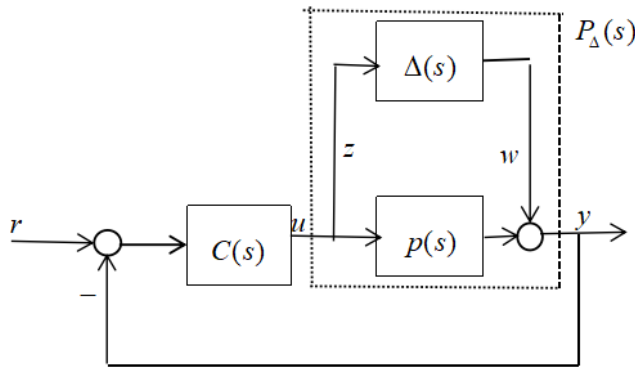


Figure 1 (1)

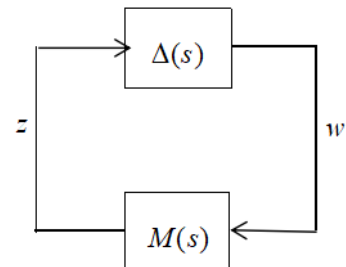


Figure 1 (2)

Figure 1: Schematic representation of a CCW I-O system within a feedback loop.

Through analysis, it is not difficult to see that Figure 1 (1) can be equivalently transformed into Figure 1 (2) through loop transformation, where $M(s) = -C(s)(I + P(s)C(s))^{-1}$. It is worth pointing out that $\Delta(s)$ is not only the relative order 2, but also it possesses a CCW I-O property. At this point, the passive system control theory is no longer applicable. This necessitates the emergence of the interconnection stability theory for CCW I-O systems.

3 Research Motivation

The CCW I-O property, as the nonlinear counterpart of the negative-imaginary property, offers a significant advantage: it accommodates systems with a relative degree of 2, whereas passivity is restricted to relative degrees of 0 or 1. If a system can be transformed into a CCW I-O system under certain appropriate conditions, then the CCW I-O theory can be utilized to analyze and synthesize it, especially for systems with a relative degree of 2. This capability is crucial for analyzing systems where passivity theory fails. The one central question is: under what circumstances can a system possess the CCW I-O property? The other is: when is the relative degree of a CCW I-O system 2?

Reference [34] provides a sufficient criterion for the relative degree of a nonlinear passive system to be equal to 1 when it is at the equilibrium point. The specific content is as follows:

Lemma 1: Assuming that system Σ has passive characteristics. If

$$\text{rank } g(0) = \text{rank } \frac{\partial h}{\partial x}(0) = n,$$

then the system will exhibit a relative degree of $\{1,1,\dots,1\}$ at the position of point $x = 0$.

Meanwhile, for the study of the feedback passivity property, Haddad et al. [34] give the following result:

Theorem 1 Consider a specific nonlinear system, denoted as Σ . Set the condition that $g(L_g h)^{-1}$ has the property of complete congruence. In this case, Σ will exhibit a feedback passive property iff Σ exhibits a relative degree of $\{1,1,\dots,1\}$ at the $x = 0$ and meets the requirement of weak minimum phase.

Reference [35] uses a specific algorithm to transform the nonlinear system in the vicinity of the origin into a system with the following expression:

$$\begin{aligned} \dot{z} &= f_1(z, \xi) \\ \dot{\xi}_3 &= \xi_1 \\ \dot{\xi}_2 &= b_2(z, \xi)u + a_2(z, \xi) \\ \dot{\xi}_1 &= b_1(z, \xi)u + a_1(z, \xi) \\ y &= [\xi_1^T, \xi_2^T]^T. \end{aligned}$$

When a_1, a_2, b_1 and b_2 are specific functions, and $[b_1(z, \xi)^T, b_2(z, \xi)^T]$ is invertible, a feedback control law is as follows:

$$u = \begin{bmatrix} b_1(z, \xi) \\ b_2(z, \xi) \end{bmatrix}^{-1} \left(v - \begin{bmatrix} a_1(z, \xi) + \left(\frac{\partial V(\xi_1)}{\partial \xi_1}\right)^T \\ a_2(z, \xi) + \left(\frac{\partial V(\xi_1)}{\partial \xi_1}\right)^T + \lambda \xi_3 \end{bmatrix} \right),$$

where v is a new input, $\lambda \geq 0$ is a number, such that the nonlinear system has the CCW I-O property.

As a further extension of the theory of passivity in terms of depth and breadth, this paper focuses on the study of nonlinear CCW I-O systems and provides sufficient criteria for determining a relative degree of 2 when the system is in equilibrium. Meanwhile, inspired by

the research results on feedback passivation, we have successfully constructed the necessary and sufficient conditions for nonlinear systems to possess feedback CCW I-O characteristics through scientific methods and rigorous deduction.

4 Preliminaries

Essential definitions are recalled below.

Definition 2[34]: For smooth vector fields, $f, g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are defined as continuously differentiable functions. The specific definition of the Lie bracket composed of f and g is as follows: $-[f(x), g(x)] = -ad_f g(x) \triangleq \frac{\partial f}{\partial x} g(x) - \frac{\partial g}{\partial x} f(x)$.

The zero order and higher-order Lie brackets are recursively defined as follows:

$$ad_f^0 g(x) \triangleq g(x), \quad ad_f^k g(x) \triangleq [f(x), ad_f^{k-1} g(x)],$$

where $k \geq 1$. The specific definition of the Lie derivative for $V: \mathbb{R}^n \rightarrow \mathbb{R}$ along the vector field of $f(x)$ is as follows:

$$L_f V(x) \triangleq \frac{\partial V}{\partial x} f(x).$$

And for the scalar function $V(\cdot)$, its 0th and higher-order Lie derivatives are defined recursively as follows:

$$L_f^0 V(x) \triangleq V(x), \quad L_f^k V(x) \triangleq L_f(L_f^{k-1} V(x)),$$

where $k \geq 1$.

Definition 3[34] (**Zero Dynamics**): The zero dynamics of (1) are the set

$$\{x \in \mathbb{R}^n | y(x(t)) \equiv 0\}.$$

Definition 4[34] (**Minimum Phase**): If there exists a function $V_0: \mathbb{R}^n \rightarrow \mathbb{R}$ with continuous differentiability and positive definiteness, which can be expressed in the following form:

$$0 > f(z)V_0'(z),$$

where $z \in Z \triangleq \{x \in \mathbb{R}^n | y(x(t)) \equiv 0\}$ and $\dot{z} = f(z)$, then system (1) has the minimum phase characteristic,

(Marginally Minimum Phase) If there exists a function $V_0: \mathbb{R}^n \rightarrow \mathbb{R}$ with continuous differentiability and positive definiteness, and the following condition is satisfied:

$$0 \geq f(z)V_0'(z),$$

where $z \in Z \triangleq \{x \in \mathbb{R}^n | y(x(t)) \equiv 0\}$ and $\dot{z} = f(z)$, then the nonlinear system (1) can be determined as an marginally minimum phase system,

(Weakly Minimum Phase) If there exists a function $V_0: \mathbb{R}^{n-m} \rightarrow \mathbb{R}$ with continuous differentiability and positive definite characteristics, then the nonlinear system (1) can be identified as a weakly minimum phase system, in which case the following relationship is satisfied:

$$0 \geq f_0(z)V_0'(z),$$

where $z \in Z \triangleq \{x \in \mathbb{R}^{n-m} | y(x(t)) \equiv 0\}$ and $\dot{z} = f_0(z)$.

Remark 1 Through the analysis of the above definition, it can be found that if the n -dimensional zero dynamics exhibit asymptotic stability, then the system can be judged as a minimum phase system; When the n -dimensional zero dynamics reach the Lyapunov stability condition, this system belongs to the marginally minimum phase system; When the $n-m$ dimensional zero dynamics satisfy the Lyapunov stability requirement, the corresponding system is a weakly minimum phase system.

Definition 5[34] (**Relative Degree**): If a neighborhood \mathcal{D}_0 belonging to x_0 can be found, such that for any given $x \in \mathcal{D}_0$,

$$L_{g_i} L_f^k h_j(x) = 0, 1 \leq i, j \leq n, 0 \leq k < r_j - 1$$

and the matrix

$$L(x) \triangleq \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \dots & L_{g_n} L_f^{r_1-1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_n-1} h_n(x) & \dots & L_{g_n} L_f^{r_n-1} h_n(x) \end{bmatrix}$$

is non-singular, where $g_i, i = 1, \dots, n$ are the n column vectors of the matrix $g, h_j, j = 1, \dots, n$ are the n components of the vector h ,

$$L_f h_j(x) \triangleq f(x) h_j'(x), \quad j \in \{1, 2, \dots, n\},$$

$$L_f^k h_j(x) \triangleq L_f(L_f^{k-1} h_j(x)), \quad 2 \leq k < r_j - 1, \quad 1 \leq j \leq n,$$

$$h_j(x) L_f^0 \triangleq h_j(x), \quad j \in \{1, 2, \dots, n\},$$

then system (1) will have the characteristic of relative degree $\{r_1, r_2, \dots, r_n\}$ at the position of point x_0 .

Remark 2 If a neighborhood \mathcal{D}_0 belonging to x_0 can be found, and for any $x \in \mathcal{D}_0$, the following conditions:

$$L_g L_f^k h(x) = 0, \quad 0 \leq k < r - 1$$

and

$$L(x) = L_g L_f^{r-1} h(x) \neq 0$$

must be satisfied, then for a single input single output (SISO) system, it will have a relative degree r at point x_0 .

Remark 3 For a multiple input multiple output (MIMO) system, when its relative degree $\{r_1, r_2, \dots, r_m\} = \{1, 1, \dots, 1\}$, matrix $L(x)$ exhibits non-singular characteristics for any given $x \in \mathcal{D}_0$, where

$$L(x) \triangleq \begin{bmatrix} L_{g_1} h_1(x) & \dots & L_{g_m} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} h_m(x) & \dots & L_{g_m} h_m(x) \end{bmatrix}.$$

Remark 4 For a multiple input multiple output (MIMO) system, when its relative degree $\{r_1, r_2, \dots, r_m\} = \{2, 2, \dots, 2\}$, the matrix $L(x)$ has non-singular properties for any $x \in \mathcal{D}_0$, where

$$L(x) \triangleq \begin{bmatrix} L_{g_1} L_f h_1(x) & \dots & L_{g_m} L_f h_1(x) \\ \vdots & \dots & \vdots \\ L_{g_1} L_f h_m(x) & \dots & L_{g_m} L_f h_m(x) \end{bmatrix}.$$

Definition 6[34] (**Distribution**): $D: \mathcal{D} \rightarrow \mathbb{R}^k$ is a k -dimensional distribution, where $\mathbb{R}^k \subset \mathbb{R}^n$, if for $\forall x \in \mathcal{D}$, there exist smooth vector fields $f_1(x), \dots, f_k(x)$ such that $D(x) = \text{span}\{f_1(x), \dots, f_k(x)\}$ is a linearly independent set.

Definition 7[34] (**Involutive**): A k -dimensional distribution $D(x) = \text{span}\{f_1(x), \dots, f_k(x)\}$, $x \in \mathcal{D}$ is involutive if for all $f_i(x), f_j(x) \in D(x)$, we have $[f_i(x), f_j(x)] \in D(x)$, where $i \neq j$, $i, j = 1, \dots, k$.

Example: Let $D(x) = \text{span}\{f_1(x), f_2(x)\}$, $\mathcal{D} = \mathbb{R}^4$, where

$$f_1(x) = \begin{bmatrix} x_1 \\ 1 \\ 0 \\ x_3 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} -\exp(x^2) \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since

$$[f_1(x), f_2(x)] = \frac{\partial f_2}{\partial x} f_1(x) - \frac{\partial f_1}{\partial x} f_2(x) = 0, \quad \text{rank}[f_1(x), f_2(x), [f_1(x), f_2(x)]] = 2,$$

it can be concluded that the distribution $D(x)$, $x \in \mathcal{D}$ is involutive.

5 Relative Degree of CCW I-O Systems and Their (Weakly) Minimum Phase Property

This section establishes sufficient conditions for a CCW I-O system to have a relative degree of $\{2, 2, \dots, 2\}$ and demonstrates that such systems are weakly minimum phase.

Theorem 2. Suppose the system (1) possesses the CCW I-O property. If $g(x) \nabla h(x) = I$, and the rank $\nabla f|_{x=0} = n$, then the nonlinear system has a relative degree of $\{2, 2, \dots, 2\}$ at $x = 0$.

Proof: When $i = 1, 2, \dots, n$,

$$L_{g_i} L_f h_1(x) = \left[\frac{\partial^2 h_1}{\partial x^2} f + \frac{\partial h_1}{\partial x} \frac{\partial f}{\partial x} \right] g_i(x).$$

At $x = 0$, we have

$$L_{g_i} L_f h_1(0) = \left(\frac{\partial h_1}{\partial x} \frac{\partial f}{\partial x} \right) \Big|_{x=0} g_i(0) \triangleq A_1 g_i(0);$$

Similarly, $L_{g_i} L_f h_i(0) = A_i g_i(0)$ ($i = 1, \dots, n$). Then, at $x = 0$,

$$\begin{aligned}
 L(0) &\triangleq \begin{bmatrix} A_1 g_1(0) & \cdots & A_1 g_n(0) \\ \vdots & \cdots & \vdots \\ A_n g_1(0) & \cdots & A_n g_n(0) \end{bmatrix} \\
 &= \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} [g_1(0) \quad g_2(0) \quad \cdots \quad g_n(0)] \\
 &= (\nabla h \nabla f(x))|_{x=0} g(0).
 \end{aligned}$$

Because system (1) has the CCW I-O property, based on the definition, we know that there exist continuously differentiable functions $V(x) \geq 0$ ($V(0) = 0$), $L(x)$, and $W(x)$ satisfying:

$$\begin{aligned}
 \frac{1}{2} g(x)^T \nabla V(x) &= \frac{1}{2} f(x) \nabla h(x) - W(x)^T L(x), \\
 f(x) \nabla V(x) &= -L(x)^T L(x), \\
 -g(x)^T \nabla h(x) &= -W(x)^T W(x).
 \end{aligned}$$

When $g(x) \nabla h(x) = I$, we have $\nabla h(x) = (g(x)^T)^{-1}$; also, from $\text{rank} \nabla f(x)|_{x=0} = n$, we know $|\nabla f(x)|_{x=0}| \neq 0$.

Thus, $(g(0)^T)^{-1} \nabla f|_{x=0} (g(0)^T)$ is non-singular. Furthermore, $L(0)$ is non-singular. From the definition of relative degree, the nonlinear system (1) has a relative degree of $\{2, 2, \dots, 2\}$ at $x = 0$.

Remark 5 For multiple input multiple output (MIMO) systems that are linear and have parameters that do not change over time (LTI):

$$\begin{cases} \dot{x} = Ax + Bu \\ y = B^{-1}x \end{cases}$$

has the CCW I-O property, and $\text{rank} A = n$, we know this linear system has a relative degree of $\{2, 2, \dots, 2\}$ at $x = 0$.

Theorem 3 Considering the given nonlinear system (1), if the system has (strictly) CCW I-O characteristics for a continuously differentiable and positively definite function V_s , then the system can be determined as a weakly minimum phase (or minimum phase) system.

Proof: Given that the system exhibits CCW I-O characteristics for a continuously differentiable and positively definite function V_s , it is possible to find $W(x)$ and $L(x)$ that satisfy the following specific conditions:

$$\begin{aligned}
 -\frac{1}{2} g(x)^T \nabla V_s(x) &= W(x)^T L(x) - \frac{1}{2} \nabla h(x) f(x), \\
 W(x)^T W(x) &= g(x)^T \nabla h(x), \\
 -L(x)^T L(x) &= \nabla V_s(x) f(x).
 \end{aligned}$$

According to the definition of zero dynamics, when $h(x) \equiv 0$ ($x \in \mathbb{R}^n$), we have $W \equiv 0$. So $g(x)^T \nabla V_s \equiv 0$.

$$\dot{V}_s(x) = \nabla V_s^T f(x) \leq u^T \dot{y} \equiv 0.$$

Therefore, along $\mathcal{D} = \{x \in \mathbb{R}^{n-m} | h(x) \equiv 0\}$, we have $\dot{V}_s(x) \leq 0$. So the system is weakly minimum phase.

Similarly, when the system has the strict CCW I-O property with respect to the positive definite function V_s , along $\mathcal{D} = \{x \in \mathbb{R}^{n-m} | h(x) \equiv 0\}$ we have $0 > \dot{V}_s(x)$. So the system is minimum phase.

Table 1 provides sufficient conditions for the nonlinear CCW I-O system to have a relative order of $\{2, 2, \dots, 2\}$ at the origin and minimum phase property.

Table 1: Sufficient Conditions for Relative Order $\{2, 2, \dots, 2\}$ at the Origin and minimum phase

Condition number	Condition Description	Theorem citation
1	System (1) has CCW I-O attribute	Theorem 2
2	$g(x) \nabla h(x) = I$	Theorem 2
3	$\text{rank } \nabla f _{x=0} = n$	Theorem 2
4	There exists a continuously differentiable and positively definite function $V_s(x)$ that satisfies specific conditions for the system	Theorem 3
Result	System (1) has a relative order $\{2, 2, \dots, 2\}$ at $x=0$ and is a weakly minimum phase (or minimum phase) system	Theorem 2, Theorem 3

6 Feedback CCW I-O Property

The corresponding control structure is shown in **Figure 2**. We now study the feedback CCW I-O property of nonlinear systems. First, we present a lemma.

Lemma 2 [34] Assume system (1) is a (weakly) minimum phase system with relative degree $\{r_1, \dots, r_n\}$. If the vector fields

$$X_j^k = \text{ad}_f^{k-1} g_j, \quad 1 \leq j \leq n, \quad 1 \leq k \leq r_j$$

are complete and involutive, then there exists a global diffeomorphism $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that there exists a coordinate transformation

$$\begin{bmatrix} y \\ z \end{bmatrix} \triangleq T(x) = Z,$$

and under this coordinate transformation, system (1) is equivalent to

$$\dot{Z} = \begin{bmatrix} L_f h(x) \\ r(z, y)y + f_0(z) \end{bmatrix} + \begin{bmatrix} L_g h(x) \\ 0 \end{bmatrix} u.$$

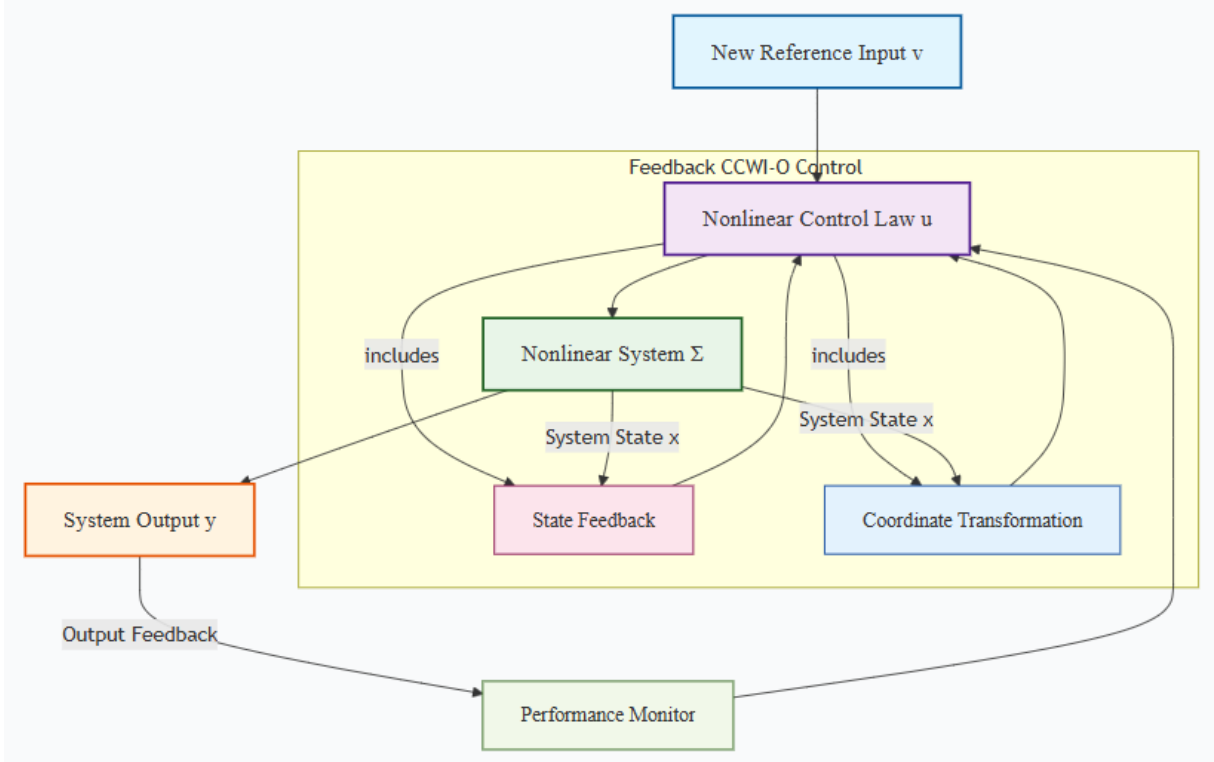


Figure 2: Block diagram of the feedback CCW I-O control structure.

Theorem 4 Given the system represented by (1), $X_j^1 = g_j$ and $X_j^2 = ad_f g_j$ are complete and involutive. If there exists $\beta(x) \neq 0$ satisfying $g(x)\beta(x)\nabla h(x) = I$, and $rank \nabla f(x)|_{x=0} = n$, then the nonlinear system is feedback CCW I-O if and only if the system has a relative degree of $\{2, 2, \dots, 2\}$ at $x = 0$ and is weakly minimum phase.

Proof: Necessity. If the system is feedback CCW I-O, then there exists $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that when $u = \alpha(x) + \beta(x)v$, for specific positive definite storage functions, nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u = f(x) + g(x)[\alpha(x) + \beta(x)v] \\ &\triangleq \tilde{g}(x)v + \tilde{f}(x) \\ y &= h(x) \end{aligned}$$

exhibits CCW I-O characteristics.

When $\nabla h(x)\tilde{g}(x) = g(x)\beta(x)\nabla h(x) = I$, and $rank \nabla f(x)|_{x=0} = n$, by Theorem 2 and Theorem 3, the system after feedback control u has a relative degree of $\{2, 2, \dots, 2\}$ at $x = 0$ and is weakly minimum phase.

From Lemma 2, there exists infinitely differentiable functions $f_0: \mathbb{R}^{n-m} \rightarrow \mathbb{R}^{n-m}$, $r: \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^{(n-m) \times m}$ and a diffeomorphism \mathcal{T} such that

$$\begin{bmatrix} y \\ z \end{bmatrix} \triangleq T(x) = Z,$$

and

$$\dot{Z} = \begin{bmatrix} L_{\tilde{f}}h(x) \\ f_0(z) + r(z, y)y \end{bmatrix} + \begin{bmatrix} L_{\tilde{g}}h(x) \\ 0 \end{bmatrix} v$$

$$\begin{aligned}
 &= \begin{bmatrix} \frac{\partial h}{\partial x} g(x)\beta(x)v + \frac{\partial h}{\partial x} [f + g(x)\alpha(x)] \\ r(z, y)y + f_0(z) \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial x} g(x)[\alpha(x) + \beta(x)v] \\ r(z, y)y + f_0(z) \end{bmatrix} \\
 &= \begin{bmatrix} L_g h(x) \\ 0 \end{bmatrix} u + \begin{bmatrix} L_f h(x) \\ r(z, y)y + f_0(z) \end{bmatrix}.
 \end{aligned}$$

From the relevant analysis, it can be clarified that the original system essentially belongs to the static feedback transformation form of feedback control system, which has a key characteristic of being able to fully preserve the system's relative degree and zero dynamic information. Based on this, for the nonlinear system (1), when in the state of $x = 0$, its relative degree is expressed as $\{2, 2, \dots, 2\}$, and the system can be determined as a weakly minimum phase system.

Sufficiency. From the conditions, the nonlinear system (1) has a relative degree of $\{2, 2, \dots, 2\}$ at $x = 0$ and is weakly minimum phase. By Lemma 2, system (1) is equivalent to

$$\begin{aligned}
 \dot{y}(t) &= L_g h(x)u(t) + L_f h(x), t \geq 0, y(0) = y_0, \\
 z(t) &= r(z(t), y(t))y(t) + f_0(z(t)), \quad z(0) = z_0.
 \end{aligned}$$

From this, it can be inferred that the above system exhibits a relative degree of $\{2, 2, \dots, 2\}$ at the $x = 0$ position and is classified as a weakly minimum phase system. According to the definition of weakly minimum phase system, there must exist a continuously differentiable and positively definite function $V_0(z) (z \in \mathbb{R}^{n-m})$, which satisfies the following conditions:

$$V_0'(z)f_0(z) \leq 0.$$

Further, we get

$$\dot{V}_0(z) \leq V_0'(z)r(z, y)y.$$

If matrix $v^T L_g h(x) - 2x^T g(x)$ has reversible mathematical properties, then the corresponding feedback control law

$$u = [v^T L_g h(x) - 2x^T g(x)]^{-1} [V_0'(z)r(z, y)y + 2x^T f(x) - (L_f h(x))v]$$

and positively definite function $V_s(z, x) = V_0(z) + x^T x$ can be selected. It can be concluded that $\dot{V}_s(z, x) \leq \dot{y}^T v$. Therefore, the nonlinear system (1) has the feedback CCW I-O property.

Remark 6 The above Theorem is degenerated to be a linear system under specific conditions, such as the following linear time invariant (LTI) system

$$\begin{cases} \dot{x} = Bu + Ax \\ y = Cx. \end{cases}$$

When $\beta(x) = (CB)^{-1}$ and $\text{rank}(A) = n$, the system exhibits feedback CCW I-O characteristics if and only if it has a relative degree $\{2, 2, \dots, 2\}$ at the $x = 0$ position and belongs to a weakly minimum phase system.

7 Conclusion

This article delves into the relative order and feedback characteristics of nonlinear counterclockwise input-output (CCW I-O) systems, achieving a series of meaningful research results. Firstly, based on the theory of dissipation and the Lie algebra method, this paper provides sufficient conditions for the nonlinear CCW I-O system to have relative orders 2 at the origin. This achievement not only expands the application scope of existing theories, especially for systems with relative order 2, but also provides a solid theoretical foundation for subsequent research. Secondly, this article proves that the strict CCW I-O system has the minimum phase characteristic. In terms of feedback characteristics, this paper derives the necessary and sufficient conditions for nonlinear systems to have feedback CCW I-O characteristics. Specifically, the system needs to have a relative order of 2 at the origin and be a weakly minimum phase system. This condition not only provides a clear criterion for determining whether the system has feedback CCW I-O characteristics, but also provides a feasible path for achieving CCW I-O characteristics through state feedback. These achievements have broad application prospects in engineering fields such as flexible structures and nano-positioning, providing effective tools and methods for the analysis and design of related systems. In order to verify the correctness and practicality of the theoretical results, this article has validated them through typical examples.

Looking ahead, there are still many directions worth exploring in depth in this study. Firstly, the current research has strict limitations on the conformity condition, and future work will focus on relaxing this condition to expand the applicability of theoretical results. Secondly, as an important means of control system design, the application of output feedback strategy in CCW I-O system has not been fully studied. Therefore, exploring the application of output feedback strategies in CCW I-O systems will be another important direction for future research. Through continuous in-depth research and improvement of CCW I-O theory, we are expected to provide more effective and flexible methods for stability analysis and controller design of more complex systems.

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