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Ergodic Properties and Probability Distribution Dynamics of a Stochastic SEIR Model with Recovery Rate

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SUMMARY: *This paper examines the dynamics of a stochastic SEIR model, proving the existence of a unique global positive solution. Conditions for an ergodic stationary distribution are determined using Lyapunov functions. An exact expression for the probability density function is derived. The threshold for persistence versus extinction is identified, and numerical simulations confirm the analysis.*

KEYWORDS: *SEIR model; ergodic stationary distribution; probability density function.*

1 Introduction

Epidemiology examines disease transmission patterns to identify the underlying causes of outbreaks. Some models describe the dynamics of disease using fractional differential equation systems(FDES)[1-4]. In the study done by Ricardo Almeida [4], a modified fractional SEIR model incorporating treatment is developed to an-alyze the progression of specific infectious diseases. The fractional SEIR model in [4] is as follows:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}s(t)=b^{\alpha}-\beta^{\alpha}i(t)s(t)-b^{\alpha}s(t) \\ {}^{C}D_{0^+}^{\alpha}e(t)=\beta^{\alpha}i(t)s(t)-\left(\delta^{\alpha}+b^{\alpha}\right)e(t) \\ {}^{C}D_{0^+}^{\alpha}i(t)=\delta^{\alpha}e(t)-\left(\mu^{\alpha}+q^{\alpha}+b^{\alpha}\right)i(t) \\ {}^{C}D_{0^+}^{\alpha}r(t)=\left(\mu^{\alpha}+q^{\alpha}\right)i(t)-b^{\alpha}r(t), \end{cases} \tag{1.1}$$

where $s(t),e(t),i(t)$ and $r(t)$ denote the rates of susceptible, exposed, infectious, and recovered individuals at time, respectively. α is the order of equation, b is the birth and natural death rate (assumed equal), β is the disease transmission rate, σ is the incubation rate, μ is the recovery (without treatment), and q is the treatment rate.

When the fractional order α approaches 1, the model reduces to a classical deterministic epidemic model. This reduction is based on the property that the fractional d erivative ${}^{C}D_{0^+}^{\alpha}$ becomes equivalent to the standard first-order derivative when $\alpha=1$ [5]. Specifically, when $\alpha=1$, the fractional derivative ${}^{C}D_{0^+}^{\alpha}$ reduces to the standard first-order derivative, and the model (1.1) become:

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$$\begin{cases} ds(t)=\left[b-\beta i(t)s(t)-bs(t)\right]dt \\ de(t)=\left[\beta i(t)s(t)-(\delta+b)e(t)\right]dt \\ di(t)=\left[\delta e(t)-(\mu+q+b)i(t)\right]dt \\ dr(t)=\left[(\mu+q)i(t)-br(t)\right]dt. \end{cases} \tag{1.2}$$

This reduction allows the model to be more readily compared and analyzed alongside classical deterministic epidemic models, providing a more intuitive math-ematical tool for understanding the mechanisms of disease transmission.

For the model(1.2), a positive invariant set can be easily derived $\Omega=\left\{(s,e,i,r)\in R_+^4 : s,e,i,r\geq 0 \text{ and } s+e+i+r=1\right\}$, a disease-free equilibrium is $P_F=(1,0,0,0)$, the basic reproduction number in this case is $R_0=\dfrac{\rho\delta}{(\delta+b)(\mu+q+b)}$.

Ricardo Almeida [4] obtained that if $R_0<1$, the disease-free equilibrium P_F is globally asymptotically stable; if $R_0>1$, it is unstable. Besides, there is an endemic equilibrium point $P_E=\left(s^*,e^*,i^*,r^*\right)$, where $s^*=\dfrac{(\delta+b)(\mu+q+b)}{\beta\delta}, e^*=\dfrac{\mu+q+b}{\delta}i^*, i^*=\dfrac{b\left(1-s^*\right)}{\beta s^*}, r^*=\dfrac{\mu+q}{b}i^*.$

In research on infectious disease dynamics, SE (Susceptible-Exposed), SEI (Susceptible-Exposed-Infectious), and SEIR (Susceptible-Exposed-Infectious-Recovered) models are commonly used mathematical frameworks to describe how infections propagate through populations. These epidemiological frameworks are built upon classical systems of deterministic ordinary differential equations [6, 7]. However, epidemic models in real-world scenarios are ofen affected by enviromental noise, which can significantly alter disease dynamics [8, 9]. Therefore, understanding the impact of environmental noise on epidemic models has become a critical area of research in recent years.

Compared to deterministic models, stochastic models offer a more accurate prediction of future system dynamics by incorporating randomness and uncertainty [10, 11]. Stochastic differential equations (SDES) provide a higher degree of realism in modeling disease transmission, as they account for random fluctuations in environmental factors[12, 13]. Consequently, numerous researchers have incorporated stochastic elements into epidemiological frameworks to better reflect real-world transmission variability[1418].

Building upon this theoretical foundation, we model environmental noise through compartment-dependent stochastic terms, where the noise intensity scales linearly with each state variable $s(t), e(t), i(t)$, $r(t)$. This yields the stochastic counterpart of system (1.2):

$$\begin{cases} ds(t)=\left[b-\beta i(t)s(t)-bs(t)\right]dt+\sigma_1 s(t)dB_1(t), \\ de(t)=\left[\beta i(t)s(t)-(\delta+b)e(t)\right]dt+\sigma_2 e(t)dB_2(t), \\ di(t)=\left[\delta e(t)-(\mu+q+b)i(t)\right]dt+\sigma_3 i(t)dB_3(t), \\ dr(t)=\left[(\mu+q)i(t)-br(t)\right]dt+\sigma_4 r(t)dB_4(t). \end{cases} \tag{1.3}$$

where the system is driven by independent, zero-initialized Brownian motions $B_j(t)$ with $B_j(0)=0$ almost surely, the parameters $\sigma_j^2>0$ govern the fluctuation amplitudes for $j=1,2,3,4$.

The triangular structure of system (1.3) reveals that the fourth equation does not influence the first three. This permits a dimension reduction without affecting the solution properties, leading to our focus on the following subsystem:

$$\begin{cases} ds(t)=\left[b-\beta i(t)s(t)-bs(t)\right]dt+\sigma_1 s(t)dB_1(t) \\ de(t)=\left[\beta i(t)s(t)-(\delta+b)e(t)\right]dt+\sigma_2 e(t)dB_2(t) \\ di(t)=\left[\delta e(t)-(\mu+q+b)i(t)\right]dt+\sigma_3 i(t)dB_3(t), \end{cases} \quad (1.4)$$

This paper is organized as follows. In section 2, we prove that there is a unique global positive solution of system(1.4). In section 3, we adopt a stochastic Lyapunov function method to establish sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the proposed stochastic model (1.4). In section 4, by solving the corresponding

Three-dimensional Fokker-Planck equation, we obtain the expression of the density function near the stable equilibrium of the deterministic system. In section 5, we establish sufficient condition for the disease to die out.In section 6, some numerical simulations are introduce=d to demonstrate the theoretical results. Finally, some conclusions are presented to end this paper.

2 Global Positivity in Stochastic Systems: Existence and Uniqueness Properties

The investigation of stochastic dynamics necessitates prior verification of solution non-negativity. Our framework confirms through Theorem 2. 1 that system (1.4) possesses exactly one solution that remains valid throughout its temporal evolution.

Our stochastic analysis begins with the celebrated differential rule from It$\hat{O}$' s calculus.
Lemma 2.1 (See[12]) Let $X(t)$ be a homogeneous Markov process in the n-dimensional Euclidean space R^n with the stochastic differential equation

$$dX(t)=f(X)dt+\sum_{j=1}^{n}g_j(X)dB_j(t)$$

where $f\in L^1\left(R_+;R^n\right)$ and $g\in L^2\left(R_+;R^{n\times m}\right)$. Let $V\in C^{2,1}\left(R^n\times R;R\right)$ then $V\left(X(t);t\right)$ is a real valued ItÔ's process with its stochastic differential given by $dV(t)=\left[V_X(X)f(X)+\frac{1}{2}tr\left[g^T(X)V_{XX}g(X)\right]\right]dt+V_X\left(X(t)\right)dB(t)$.

Given $V\in C^{2,1}\left(R^n\times R;R\right)$, define an operator $£V=V_X(X)f(X)+\frac{1}{2}tr\left[g^T(X)V_{XX}g(X)\right]$.

where $V_X=\left\{\partial V/\partial X_1,\cdots,\partial V/\partial X_n\right\}, V_{XX}=\left(\partial^2 V/\partial X_j\partial X_k\right)_{n\times n}$ and the notation tr denotes the trace of the matrix.

The diffusion matrix of the process $X(t)$ is defined as follows: $A(x)=\left(a_{jk}\right), a_{jk}(x)=\sum_{l=1}^{t}g_l^j(x)g_l^k(x)$.

Lemma 2. 1

Next, we will prove that the stochastic model (1.4) has a unique global solution.

Theorem 2.1 Let $\left(s(0),e(0),i(0)\right)$ be an arbitrary point in the positive cone $\mathbb{R}_+^3$. Then there exists a unique stochastic process $\left(s(t),e(t),i(t)\right)$ solving (1.4) for $t\in(0,\infty)$, possessing the properties that $P\left\{(s(t),e(t),i(t))\in\mathbb{R}_+^3 \text{ for all } t>0\right\}=1$.

Theorem 2. 1

Proof Under the local Lipschitz continuity of the coefficients in system (1.4), for any initial condition $\left(s(0),e(0),i(0)\right)\in\mathbb{R}_+^3$, a pathwise unique solution $\left(s(t),e(t),i(t)\right)$ exists and remains in $\mathbb{R}_+^3$ over a maximal interval $\left[0,\tau_e\right]$, where τ_e denotes the potential explosion time [12]. To establish global existence, it suffices to prove that $\tau\to\infty$ a.s.

To establish that the solution remains finite in finite time, we begin by selecting a sufficiently large integer $k_0\geq 1$ such that the initial conditions $s(0),e(0)$ and $i(0)$ are confined to the interval $\left[k_0^{-1},k_0\right]$.

Consider the non-negative Lyapunov-type function

$$W(s,e,i)=\left(s-a_1-a_1\ln\frac{s}{a_1}\right)+a_2\left(e-1-\ln e\right)+a_3\left(i-1-\ln i\right),$$

where a_1,a_2,a_3 are positive parameters to be determined. Noting that $u-1-\ln u\geq 0$ for any $u>0$, and applying Itô's lemma to W, we derive the stochastic differential:

$$dW(s,e,i)=£\mathrm{W}(s,e,i)dt+\sigma_1\left(s-a_1\right)dB_1(t)+\sigma_2a_2\left(e-1\right)dB_2(t)+\sigma_3a_3\left(i-1\right)dB_3(t),$$

where the differential operator $£\mathrm{W}:\mathbb{R}_+^3\to\mathbb{R}$ is given by:

$$\begin{aligned}
£\mathrm{W}&=b-\beta is-bs-\frac{a_1b}{s}+\beta i+a_1b+\frac{1}{2}a_1\sigma_1^2+a_2\beta is\\
&-a_2\left(\delta+b\right)e-a_2\frac{\beta is}{e}+a_2\left(\delta+b\right)+\frac{1}{2}a_2\sigma_2^2+a_3\delta e-a_3\left(u+q+b\right)i-a_3\frac{\delta e}{i}\\
&+a_3\left(u+q+b\right)+\frac{1}{2}a_3\sigma_3^2\\
&\leq\left(1+a_1\right)b+a_2\left(\delta+b\right)+a_3\left(u+q+b\right)+\frac{1}{2}\left(a_1\sigma_1^2+a_2\sigma_2^2+a_3\sigma_3^2\right)\\
&+\left(a_2-1\right)\beta is+\left[a_1\beta-a_3\left(u+q+b\right)\right]i+\left[a_3\delta-a_2\left(\delta+b\right)\right]e.
\end{aligned}$$

Choose $a_1=\dfrac{(\delta+b)(u+q+b)}{\delta\beta},a_2=1,a_3=\dfrac{\delta+b}{\delta}$, then we have

$$£\mathrm{W}\leq\left(1+a_1\right)b+a_2\left(\delta+b\right)+a_3\left(u+q+b\right)+\frac{1}{2}\left(a_1\sigma_1^2+a_2\sigma_2^2+a_3\sigma_3^2\right):=M,$$

since the positive constant M depends neither on s,e nor i, the remaining analysis follows standard arguments analogous to those in [19] and Theorem 2. 1. For brevity, we omit these technical details. Thus, the proof is concluded. □

3 Existence of Ergodic Stationary Distribution

The primary focus of this investigation centers on establishing conditions under which the system admits a time-invariant probability measure with ergodic properties. To lay the foundation for this analysis, we first present fundamental theoretical prerequisites concerning invariant measures for stochastic dynamical systems. (A comprehensive treatment of these foundational concepts can be found in the seminal work [20]).

Lemma 3.1 (see [20]) A Markov process admits a unique ergodic invariant probability measure $\pi(\bullet)$ when the following conditions are satisfied for some bounded open domain $U \subset \mathbb{R}^n$ with sufficiently smooth boundary ∂U :

(A_1) Uniform ellipticity holds within U , i.e., there exists $K > 0$ such that the diffusion coefficients satisfy the uniform parabolicity condition:

$$\sum_{n} a_{jl}(x)\xi_j\xi_l \geq K\|\xi\|^2 \quad \forall x \in U, \forall \xi \in \mathbb{R}^n.$$

(A_2) There exists a Lyapunov-type function $V \in C^2(\mathbb{R}^n, \mathbb{R}_+)$ such that the infinitesimal generator $£V$ £V satisfies: $£V(x) < 0, \quad \forall x \in \mathbb{R}^n$ with the additional dissipativity property that $£V(x) < 1$ outside U .

Lemma 3. 1

Under these conditions, the process exhibits unique ergodicity with stationary distribution π, and satisfies the almost sure ergodic convergence:

$$P\left\{\lim_{T\to\infty}\int_0^T f\left(X\left(t\right)\right)dt = \int_{R^n} f\left(x\right)\pi\left(dx\right) = 1\right\}$$

for all initial states $x \in \mathbb{R}^n$ and any $\pi -$integrable function f .

Theorem 3.1 Assume that

$$R_0^s := \frac{b\delta\beta}{\left(b+\frac{1}{2}\sigma_1^2\right)\left(\delta+b+\frac{1}{2}\sigma_2^2\right)\left(u+q+b+\frac{1}{2}\sigma_3^2\right)} > 1,$$

then system (1.4) has a stationary distribution $\pi(\bullet)$ and the ergodicity holds.

Theorem 3. 1

Proof To establish the existence of a stationary distribution via Lemma 3. 1, we must validate its two fundamental hypotheses. The diffusion matrix associated with system (1.4) takes the form

$$A = \begin{pmatrix} \sigma_1^2 s^2 & 0 & 0 \\ 0 & \sigma_2^2 e^2 & 0 \\ 0 & 0 & \sigma_3^2 i^2 \end{pmatrix}.$$

Choosing

$$k = \min\left(s,e,i\right) \in \overline{U_n} \subset R^3\left(\sigma_1^2 s^2, \sigma_2^2 e^2, \sigma_3^2 i^2\right),$$

$$\sum_{m,n=1}^{3} a_{mn}\left(s,e,i\right)\xi_m\xi_n = \sigma_1^2 s^2 \xi_1^2 + \sigma_2^2 e^2 \xi_2^2 + \sigma_3^2 i^2 \xi_3^2 \geq k\left\|\xi\right\|^2,$$

for all state vectors $(s,e,i)^{\mathrm{T}}$ within the compact domain $\bar{U}_n = [n^{-1},n]^3 \subset \mathbb{R}_+^3$ and arbitrary direction vectors $\xi = (\xi_1,\xi_2,\xi_3)^{\mathrm{T}} \in \mathbb{R}_+^3$, the uniform ellipticity condition $\left(A_1\right)$ is satisfied. This establishes the first requirement.

The principal remaining task involves demonstrating the validity of the Lyapunov stability condition$\left(A_2\right)$.

Consider a twice continuously differentiable candidate function defined as

$$U_1 = -\ln e - c_1 \ln s - c_2 \ln i \tag{3.1}$$

Applying the stochastic differential rule (It$\hat{O}$'s calculus) to this functional yields:

$$\begin{aligned} \pounds U_1 &= \delta + b + \frac{1}{2}\sigma_2^2 + c_1\beta i + c_1\left(b + \frac{1}{2}\sigma_1^2\right) + c_2\left(u + q + b + \frac{1}{2}\sigma_3^2\right) \\ &\quad - \frac{\beta i s}{e} - \frac{c_1 b}{s} - \frac{c_2 \delta e}{i} \\ &\leq \delta + b + \frac{1}{2}\sigma_2^2 + c_1\left(b + \frac{1}{2}\sigma_1^2\right) + c_2\left(u + q + b + \frac{1}{2}\sigma_3^2\right) - 3\sqrt[3]{c_1 c_2 b\delta\beta} + c_1\beta i. \end{aligned} \tag{3.2}$$

Let

$$c_1 = \frac{b\delta\beta}{\left(b + \frac{1}{2}\sigma_1^2\right)^2\left(u + q + b + \frac{1}{2}\sigma_3^2\right)},$$

$$c_2 = \frac{b\delta\beta}{\left(b + \frac{1}{2}\sigma_1^2\right)\left(u + q + b + \frac{1}{2}\sigma_3^2\right)^2}.$$

Then

$$\begin{aligned} \pounds U_1 &\leq \delta + b + \frac{1}{2}\sigma_2^2 - \frac{b\delta\beta}{\left(b + \frac{1}{2}\sigma_1^2\right)\left(u + q + b + \frac{1}{2}\sigma_3^2\right)} \\ &\quad + \frac{b\delta\beta}{\left(b + \frac{1}{2}\sigma_1^2\right)^2\left(u + q + b + \frac{1}{2}\sigma_3^2\right)} i \\ &= -\left(\delta + b + \frac{1}{2}\sigma_2^2\right)\left(R_0^s - 1\right) + \frac{b\delta\beta i}{\left(b + \frac{1}{2}\sigma_1^2\right)^2\left(u + q + b + \frac{1}{2}\sigma_3^2\right)}, \end{aligned} \tag{3.3}$$

where

$$R_0^s := \frac{b\delta\beta}{\left(b+\frac{1}{2}\sigma_1^2\right)\left(\delta+b+\frac{1}{2}\sigma_2^2\right)\left(u+q+b+\frac{1}{2}\sigma_3^2\right)}.$$

Next, we define

$$U_2 = -\ln s - \ln i, N = s+e+i, U_3 = \frac{1}{\theta+1}N^{1+\theta},$$

where $\theta \in \left(0, \frac{b+\delta}{\delta} \wedge 2(u+q+b)/\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)$ is a sufficiently small constant.

Applying ItÔ's formula to U_2 and U_3 respectively, we have

$$£U_2 = -\frac{b}{s} + \beta i + b + \frac{1}{2}\sigma_1^2 + u + q + b + \frac{1}{2}\sigma_3^2 - \frac{\delta}{i}e. \quad (3.4)$$

$$\begin{aligned}
£U_3 &= bN^\theta - \left\{bs + \frac{b+\delta}{2\delta}e + \frac{(b+\delta)(u+q+b)}{2\delta}i\right\}N^\theta \\
&\quad + \frac{1}{2}\theta N^{\theta-1}\left[\sigma_1^2 s^2 + \sigma_2^2 e^2 + h^2\sigma_3^2 i^2\right] \\
&\le bN^\theta - \left(\frac{b+\delta}{2\delta} \wedge (u+q+b)\right)N^{\theta+1} + \frac{1}{2}\theta\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)N^{\theta+1} \\
&= bN^\theta - \left[\frac{b+\delta}{2\delta} \wedge (u+q+b) - \frac{1}{2}\theta\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)\right]N^{\theta+1} \quad (3.5) \\
&= bN^\theta - \frac{v}{2}N^{\theta+1} - \frac{v}{2}N^{\theta+1} \\
&\le bN^\theta - \frac{v}{2}N^{\theta+1} - \frac{v}{2}\left(s^{\theta+1} + e^{\theta+1} + h^{\theta+1}i^{\theta+1}\right) \\
&= J_1 - \frac{v}{2}\left(s^{\theta+1} + e^{\theta+1} + h^{\theta+1}i^{\theta+1}\right).
\end{aligned}$$

where

$$h = \frac{b+\delta}{2\delta}$$

$$v := \frac{b+\delta}{2\delta} \wedge (u+q+b) - \frac{1}{2}\theta\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right),$$

$$J_1 := \sup_{(s,e,i)\in R_+^3}\left\{bN^\theta - \frac{v}{2}N^{\theta+1}\right\} < \infty.$$

Define a C^2- function $U: R_+^3 \to R$ as follows:

$$\begin{aligned} U(s,e,i) &= \mathcal{M}£U_1 + £U_2 + £U_3 \\ &\le -\mathcal{M}\left(\delta + b + \frac{1}{2}\sigma_2^2\right)\left(R_0^s - 1\right) + \left(\mathcal{M}c_1 + 1\right)\beta i \\ &+ b + \frac{1}{2}\sigma_1^2 + u + q + b + \frac{1}{2}\sigma_3^2 + J_1 - \frac{v}{2}\left(s^{\theta+1} + e^{\theta+1} + h^{\theta+1} i^{\theta+1}\right) \\ &= -\mathcal{M}\left(\delta + b + \frac{1}{2}\sigma_2^2\right)\left(R_0^s - 1\right) + \left(\mathcal{M}c_1 + 1\right)\beta i \\ &- \frac{v}{4}\left(s^{\theta+1} + e^{\theta+1} + h^{\theta+1} i^{\theta+1}\right), \end{aligned} \tag{3.6}$$

where $\mathcal{M}$ is sufficiently large positive constant satisfying the following condition

$$-\mathcal{M}\left(\delta + b + \frac{1}{2}\sigma_2^2\right)\left(R_0^s - 1\right) + J_2 \le -2 \tag{3.7}$$

and

$$J_2 := \sup_{(s,e,i)\in R_+^3} \left\{ -\frac{v}{4}\left(s^{\theta+1} + e^{\theta+1} + h^{\theta+1} i^{\theta+1}\right) + J_1 + 2b + u + q + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 \right\} < \infty .$$

Furthermore, observe that the function $U(s,e,i)$ exhibits both continuity and asymptotic divergence toward infinity as the boundary of the positive orthant $\mathbb{R}^3$ is approached. Consequently, this functional must possess a finite infimum that is achieved at an interior point $\left(s^0, e^0, i^0\right) \in \mathbb{R}_+^3$. We subsequently construct a twice-differentiable mapping $\bar{U}: \mathbb{R}_+^3 \to \overline{\mathbb{R}}_+$ with these properties.

$$\begin{aligned} \overline{U}(s,e,i) &= U(s,e,i) - U\left(s^0, e^0, i^0\right) \\ &= \mathcal{M}U_1(s,e,i) + U_2(s,e,i) + U_3(s,e,i) - U\left(s^0, e^0, i^0\right). \end{aligned}$$

According to (3.2),(3.4) and (3.5)(3.2), we obtain

$$\begin{aligned} £\overline{U} \le &-\mathcal{M}\left(\delta + b + \frac{1}{2}\sigma_2^2\right)\left(R_0^s - 1\right) + \left(\mathcal{M}c_1 + 1\right)\beta i - \frac{b}{s} + b + \frac{1}{2}\sigma_1^2 \\ &- \frac{\delta}{i}e + u + q + b + \frac{1}{2}\sigma_3^2 + J_1 - \frac{v}{2}\left(s^{\theta+1} + e^{\theta+1} + h^{\theta+1} i^{\theta+1}\right). \end{aligned} \tag{3.8}$$

We now define a tightly bounded domain D_ε to satisfy requirement (A_2) in Lemma 3. 1. Specifically, we select a confined, topologically closed region structured as follows:

$$D_\varepsilon = \left\{ (s,e,i)^T \in R_+^3 : \varepsilon \le s \le \frac{1}{\varepsilon}, \varepsilon \le s \le \frac{1}{\varepsilon}, \varepsilon^2 \le s \le \frac{1}{\varepsilon^2} \right\},$$

and

$$\tilde{D} = \{0 < s < 1, 0 < e < 1, 0 < i < 1, 0 < s + e + i \le 1\},$$

for an arbitrarily small parameter $\varepsilon \in (0,1)$, the excluded domain $\mathbb{R}_+^3 \setminus D_\varepsilon$ admits a selection of ε sufficiently minute to ensure the subsequent criteria are satisfied:

$$-\frac{b}{\varepsilon} + F \le -1, \tag{3.9}$$

$$\varepsilon^2 \le \frac{1}{(\mathcal{M}c_1 + 1)\beta}, \tag{3.10}$$

$$-\frac{\delta}{\varepsilon} + F \le -1, \tag{3.11}$$

$$-\frac{1}{4\varepsilon^{\theta+1}} + F \le -1, \tag{3.12}$$

$$-\frac{v}{4} h^{(\theta+1)} \frac{1}{\varepsilon^{2\theta+2}} + F \le -1, \tag{3.13}$$

$$-\frac{v}{4} h^{(\theta+1)} \frac{1}{\varepsilon^{2\theta+2}} + F \le -1, \tag{3.14}$$

and

$$F := \sup_{(s,e,i)\in R_+^3} \left\{ -\frac{v}{4}\left(s^{\theta+1} + e^{\theta+1} + h^{(\theta+1)} i^{\theta+1} \right) + (\mathcal{M}c_1 + 1)\beta i + J_1 + 2b + u + q + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 \right\} < \infty.$$

To facilitate analysis, we partition the non-negative orthant's complement of D_ε into six distinct regions as follows:

$$D_{1,\varepsilon}^c = \left\{ (s,e,i) \in \tilde{D} \,\middle|\, s < \varepsilon \right\},$$

$$D_{2,\varepsilon}^c = \left\{ (s,e,i) \in \tilde{D} \,\middle|\, i < \varepsilon^2 \right\},$$

$$D_{3,\varepsilon}^c = \left\{ (s,e,i) \in \tilde{D} \,\middle|\, i \le \varepsilon^2, e > \varepsilon \right\},$$

$$D_{4,\varepsilon}^c = \left\{ (s,e,i) \in \tilde{D} \,\middle|\, s > \frac{1}{\varepsilon} \right\},$$

$$D_{5,\varepsilon}^c = \left\{ (s,e,i) \in \tilde{D} \,\middle|\, e > \frac{1}{\varepsilon} \right\},$$

$$D_{6,\varepsilon}^{c}=\left\{(s,e,i)\in\tilde{D}\,\middle|\,i>\frac{1}{\varepsilon^{2}}\right\}.$$

We now demonstrate that the differential operator £ acting on $\bar{U}(s,e,i)$ yields $£\bar{U}(s,e,i)\le -1$ throughout all six specified subregions of the state space $\mathbb{R}_{+}^{3}$.

Case 1: $(s,e,i)\in D_{1,\varepsilon}^{c}$, according to (3.8), we have

$$\begin{aligned}£\overline{U}&\le -\frac{b}{s}-\mathcal{M}\left(\delta+b+\frac{1}{2}\sigma_2^2\right)\left(R_0^s-1\right)+\left(\mathcal{M}c_1+1\right)\beta i+b+\frac{1}{2}\sigma_1^2\\&\quad-\frac{\delta}{i}e+u+q+b+\frac{1}{2}\sigma_3^2+J_1-\frac{v}{4}\left(s^{\theta+1}+e^{\theta+1}+h^{\theta+1}i^{\theta+1}\right)\\&\le -\frac{b}{s}+F\\&\le -\frac{b}{\varepsilon}+F\le -1,\end{aligned}\tag{3.15}$$

which follows from (3.9).

Case 2: $(s,e,i)\in D_{2,\varepsilon}^{c}$, by means of (3.8), we derive

$$\begin{aligned}£\overline{U}&\le -\mathcal{M}\left(\delta+b+\frac{1}{2}\sigma_2^2\right)\left(R_0^s-1\right)+\left(\mathcal{M}c_1+1\right)\beta i+J_2\\&\le -2+\left(\mathcal{M}c_1+1\right)\beta i\\&\le -1,\end{aligned}\tag{3.16}$$

which follows from (3.7) and (3.10).

Case 3: $(s,e,i)\in D_{3,\varepsilon}^{c}$, in view of (3.8), we obtain

$$\begin{aligned}£\overline{U}&\le \frac{\delta}{i}e+F\\&\le \frac{\delta}{\varepsilon}+F\\&\le -1,\end{aligned}\tag{3.17}$$

which follows from (3.11).

Case 4: $(s,e,i)\in D_{4,\varepsilon}^{c}$, by means of (3.8), we get

$$\begin{aligned}£\overline{U}&\le -\frac{v}{4}s^{\theta+1}+F\\&\le -\frac{v}{4}\frac{1}{\varepsilon^{\theta+}1}+F\\&\le -1,\end{aligned}\tag{3.18}$$

which follows from (3.12).

Case 5: $(s,e,i)\in D_{5,\varepsilon}^{c}$, according to (3.12), we have

$$\begin{aligned}\pounds\overline{U} &\le -\frac{v}{4}e^{\theta+1}+F\\ &\le -\frac{v}{4}\frac{1}{e^{\theta+}1}+F\\ &\le -1,\end{aligned}\tag{3.19}$$

which follows from (3.13).

Case 6: $(s,e,i)\in D_{6,\varepsilon}^{c}$, by means of (3.8), we have

$$\begin{aligned}\pounds\overline{U} &\le -\frac{v}{4}h^{\theta+1}i^{\theta+1}+F\\ &\le -\frac{vh^{\theta+1}}{4\varepsilon^{2(\theta+1)}}+F\\ &\le -1,\end{aligned}\tag{3.20}$$

which follows from (3.14).

Therefore, take $\varepsilon>0$ sufficiently small, then $\pounds\overline{U}(s,e,i)\le -1$, for $(s,e,i)\in D_{j,\varepsilon}^{c}=R_{+}^{3}\setminus D_{\varepsilon}$.

This verification demonstrates that requirement (A_2) in Lemma 3. 1 is satisfied.

Consequently, the stochastic system described by (1.4) admits exactly one invariant probability measure $\pi(\bullet)$ and exhibits ergodic behavior.□

4 Density Function of Stochastic SEIR Epidemic Model (1.4)

4.1 The density function of stochastic SEIR epidemic model

In this section, we aim to find the local probability density function of system (1.4)

Applying the logarithmic transformations $x_1=\ln s, x_2=\ln e, x_3=\ln i$, It $\hat{O}'s$ calculus combined with system (1.4) yields:

$$\begin{cases}dx_1=\left(be^{-x_1}-\beta^{\alpha}e^{-x_3}-b\right)dt+\sigma_1 dB_1(t),\\ dx_2=\left(\beta e^{x_1-x_2+x_3}-(\delta+b)\right)dt+\sigma_2 dB_2(t),\\ dx_3=\left(\delta e^{x_2-x_3}-(u+q+b)\right)dt+\sigma_3 dB_3(t).\end{cases}\tag{4.1}$$

Assume that $R_0^s>1$, there is the quasi-infected steady state $E^*\left(s^*,e^*,i^*\right)=\left(e^{x_1^*},e^{x_2^*},e^{x_3^*}\right)\in R_+^3$ determined by the following equations:

$$\begin{cases} be^{-x_1^*} - \beta e^{-x_3^*} - b = 0, \\ \beta e^{x_1^* - x_2^* + x_3^*} - (\delta + b) = 0, \\ \delta e^{x_2^* - x_3^*} - (u + q + b) = 0. \end{cases} \tag{4.2}$$

It is obvious that

$$s^* = \frac{(\delta^\alpha + b^\alpha)(u^\alpha + q^\alpha + b^\alpha)}{\beta^\alpha \delta^\alpha}, e^* = \frac{u^\alpha + q^\alpha + b^\alpha}{\delta^\alpha} i^*, i^* = \frac{\delta\beta - b(\delta + b)(u + q + b)}{\beta(\delta + b)(u + q + b)}$$

demonstrate exact correspondence with the asymptotically stable endemic fixed point (s^*, e^*, i^*) of the deterministic system (1.4).

Performing a coordinate shift $(y_1, y_2, y_3) = (x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*)$, where $x_1^* = \ln s^*, x_2^* = \ln e^*, x_3^* = \ln i^*$ represent logarithmic equilibrium coordinates, the linear approximation of system (1.4) takes the form:

$$\begin{cases} dy_1 = (-a_{11}y_1 - a_{13}y_3)dt + \sigma_1 dB_1(t), \\ dy_2 = (a_{21}y_1 - a_{21}y_2 + a_{21}y_3)dt + \sigma_2 dB_2(t), \\ dy_3 = (a_{32}y_2 - a_{32}y_3)dt + \sigma_3 dB_3(t). \end{cases} \tag{4.3}$$

where

$$a_{11} = be^{x_1^*}, a_{13} = \beta e^{x_3^*}, a_{21} = \beta e^{x_1^* - x_2^* + x_3^*}, a_{32} = \delta e^{x_2^* - x_3^*}.$$

We see that $a_{11} > 0, a_{13} > 0$ and $a_{21} > 0$.

The following theorem gives the results about the local probability density function of system (1.4).

Theorem 4.1 When $R_0^s > 1$, the system exhibits a unique log-normal stationary distribution $\Phi(s,e,i)$ concentrated near E^* for all initial conditions $(s(0), e(0), i(0)) \in \mathbb{R}_+^3$, expressible as:

$$\Phi(s,e,i) = (2\pi)^{-\frac{3}{2}} (\Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}\left(\ln\frac{s}{s^*}, \ln\frac{e}{e^*}, \ln\frac{i}{i^*}\right)(\Sigma)^{-1}\left(\ln\frac{s}{s^*}, \ln\frac{e}{e^*}, \ln\frac{i}{i^*}\right)^T},$$

where the covariance matrix Σ is positive definite,and the specific form of Σ is given as follows:

(i) if $m \neq 0$,then

$$\Sigma = \rho_1^2 J_1^{-1} \Sigma_0 (J_1^{-1})^T + \rho_2^2 (J_3 J_2)^{-1} \Sigma_0 \left[(J_3 J_2)^{-1}\right]^T + \rho_3^2 (J_6 J_5 J_4)^{-1} \Sigma_0 \left[(J_6 J_5 J_4)^{-1}\right]^T.$$

(ii) if $m = 0$,then

$$\Sigma = \rho_1^2 J_1^{-1} \Sigma_0 \left(J_1^{-1}\right)^T + \rho_2^2 \left(J_3 J_2\right)^{-1} \Sigma_0 \left[\left(J_3 J_2\right)^{-1}\right]^T + \rho_3^2 \left(J_7 J_5 J_4\right)^{-1} \theta_0 \left[\left(J_7 J_5 J_4\right)^{-1}\right]^T,$$

where

$$J_1 = \begin{pmatrix} -a_{13}a_{32} & a_{32} - a_{21}a_{32} & a_{21}a_{32} + a_{32} \\ 0 & a_{32} & -a_{32} \\ 0 & 0 & 1 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} -a_{13}a_{32} & a_{13}\left(a_{11} + a_{32}\right) & a_{11}^2 \\ 0 & -a_{13} & -a_{11} \\ 0 & 0 & 1 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$J_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{a_{21}}{a_{13}} & 1 \end{pmatrix},$$

$$J_6 = \begin{pmatrix} -ma_{13} & -m\left(a_{11} + a_{21}\right) & a_{21}^2 \\ 0 & m & -a_{21} \\ 0 & 0 & 1 \end{pmatrix},$$

$$J_7 = \begin{pmatrix} -a_{13} & -a_{11} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho_1 = a_{21}a_{32}\sigma_1, \rho_2 = -a_{13}a_{32}\sigma_2, \rho_3 = -a_{13}m\sigma_3, \rho_3 = -a_{13}\sigma_3, m = \frac{a_{21}a_{32} + a_{21}^2 - a_{11}a_{21}}{a_{13}},$$

$$\tau_1 = a_{11} + a_{21} + a_{32},$$

$$\tau_2 = a_{11}\left(a_{21} + a_{32}\right),$$

$$\tau_3 = a_{11}a_{21}a_{32},$$

$$\theta_0 = \begin{pmatrix} \frac{1}{2l_1} & 0 & 0 \\ 0 & \frac{1}{2l_1 l_2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$l_1 = a_{11} + a_{32},$$

$$l_2 = a_{13}a_{32}.$$

Theorem 4. 1

In order to prove this theorem, we need to introduce an important definition and two necessary lemmas.

Definition 4.1 (See [21]) The characteristic polynomial of the square matrix A_n is called a Hurwitz matrix if and only if A_n has all negative real-part eigenvalue, that is,

$$H_k = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ 1 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_k \end{vmatrix} > 0, k = 1, 2, \ldots, n,$$

where the complementary definition is $a_j = 0, j > n.$ Additionally, the corresponding necessary conditions for A_n to be Hurwitz matrix are as follows

(i) $a_j > 0,\ j = 1, 2, \ldots n;$

(ii) $a_j a_{j+1} > a_{j-1}a_j,\ j = 1, 2, \ldots n-1 (a_0 = 1).$

Lemma 4.1 (See [22]) For the algebraic equation $H_0^2 + A_0\Sigma_0 + A_0 C_0^T = 0,$ where $H_0 = diag(1, 0, 0),$

$$A_0 = \begin{pmatrix} -\tau_1 & -\tau_2 & -\tau_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

if $\tau_1 > 0, \tau_3 > 0$ and $\tau_1\tau_2 - \tau_3 > 0$, then Σ_0 is positive definite, where

$$\Sigma_0 = \begin{pmatrix} \frac{\tau_2}{2(\tau_1\tau_2 - \tau_3)} & 0 & -\frac{1}{2(\tau_1\tau_2 - \tau_3)} \\ 0 & 1 & 0 \\ -\frac{1}{2(\tau_1\tau_2 - \tau_3)} & 0 & \frac{\tau_1}{2\tau_3(\tau_1\tau_2 - \tau_3)} \end{pmatrix}.$$

here C_0 in this form is called the standard R_1 matrix.

Lemma 4. 1

Lemma 4.2 (See [22]) For the algebraic equation $H_0^2 + B_0\theta_0 + \theta_0 B_0^T = 0,$ where

$$H_0 = diag(1,0,0),$$

$$B_0 = \begin{pmatrix} -l_1 & -l_2 & -l_2 \\ 1 & 0 & 0 \\ 0 & 0 & -a_{21} \end{pmatrix},$$

if $l_1 > 0$ and $l_2 > 0$, then θ_0 is semi-positive definite which follows

$$\theta_0 = \begin{pmatrix} \frac{1}{2l_1} & 0 & 0 \\ 0 & \frac{1}{2l_1 l_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here B_0 in this form is called the standard R_2 matrix .

Lemma 4. 2

4.2 The proof of theorem 4.1

The following argument provides a rigorous verification of Theorem 4. 1.

Proof In (4.3), let $Y = \{y_1, y_2, y_3\}^T$, $H = diag(\sigma_1, \sigma_2, \sigma_3)$,then $dY = AYdt + HdB(t)$, where

$$A = \begin{pmatrix} -a_{11} & 0 & -a_{12} \\ a_{21} & -a_{21} & a_{21} \\ 0 & a_{32} & -a_{32} \end{pmatrix}.$$

According to [23], the density function $\Phi(Y) = \Phi(y_1, y_2, y_3)$ of the quasi-stationary distribution of system (4.3) around the origin point can be approximated by the following three-dimensional Fokker-Plank equation

$$\frac{\partial}{\partial y_1}\left[\left(-a_{11}y_1-a_{13}y_3\right)\Phi\right]+\frac{\partial}{\partial y_2}\left[\left(a_{21}y_1-a_{21}y_2+a_{21}y_3\right)\Phi\right]$$
$$+\frac{\partial}{\partial y_3}\left[\left(a_{32}y_2-a_{32}y_3\right)\Phi\right]-\sum_{j=1}^{3}\frac{\sigma_j^2}{2}\frac{\partial^2\Phi}{\partial x_j^2}=0, \tag{4.4}$$

which can be approximated by a Gaussian distribution

$$\Phi\left(Y\right)=c\exp\left\{-\frac{1}{2}YQY^T\right\},$$

where the normalization constant $c>0$ ensures $\int\Phi(Y)dY=1$ over $\mathbb{R}_+^3$, and Q denotes a real-valued symmetric matrix.

Substituting these results into (4.1), we can get the constant $c=\left(2\pi\right)^{-\frac{3}{2}}\left|Q\right|^{-\frac{1}{2}}$ and Q satisfies the following algebraic equation

$$QH^2Q^T+QA+A^TQ=0. \tag{4.5}$$

If Q is positive definite, let $Q^{-1}=\Sigma$, then

$$H^2+A\Sigma+\Sigma A^T=0. \tag{4.6}$$

The finite superposition principle [23] decomposes equation (4.6) into three constituent relations

$$H_j^2+A\Sigma_j+\Sigma_jA^T=0,\left(j=1,2,3\right)$$

where $H_1=diag\left(\sigma_1,0,0\right),H_2=diag\left(0,\sigma_2,0\right),H_3=diag\left(0,0,\sigma_3\right)$, $\Sigma=\Sigma_1+\Sigma_2+\Sigma_3$, $H^2=H_1^2+H_2^2+H_3^2$.

Analyzing the positive definiteness of Σ necessitates prior investigation of the spectral properties of A through examination of its characteristic polynomial

$$\varphi_A\left(\lambda\right)=\lambda^3+\tau_1\lambda^2+\tau_2\lambda+\tau_3=0, \tag{4.7}$$

where τ_1,τ_2 and τ_3 are positive constants and satisfying

$$\tau_1=a_{11}+a_{21}+a_{32},$$
$$\tau_2=a_{11}\left(a_{21}+a_{32}\right),$$
$$\tau_1=a_{13}a_{21}a_{32}.$$

The explicit forms of s^*,e^*,i^* under the condition $R_0^s>1$ guarantee $\tau_1>0,\tau_2>0,\tau_1\tau_2-\tau_3>0$. Consequently, all eigenvalues of the characteristic polynomial (4.7) possess negative real parts, confirming A is Hurwitz-stable.

As fundamental similarity invariants, the coefficients τ_1,τ_2,τ_3 of $\varphi_A\left(\lambda\right)$ maintain their values through matrix similarity transformations, which guarantees uniqueness for the standard R_1 form of matrix A.

The explicit construction of Σ and its positive definiteness verification proceeds through three sequential stages:

(i) prove the matrix Σ_1 is positive definiteness;

(ii) verify the matrix Σ_2 is positive definiteness conditions ;

(iii) show the matrix Σ_3 is at lest semi-positive definiteness.

Step 1. Consider the algebraic equation

$$H^2 + A\Sigma_1 + \Sigma_1 A^T = 0 \tag{4.8}$$

where $H_1 = diag\left(\sigma_1, 0, 0\right)$. Let $A_1 = J_1 A J_1^{-1}$, then the standardized transformation matrix J_1 is given by

$$J_1 = \begin{pmatrix} a_{21}a_{32} & a_{32} - a_{21}a_{32} & a_{21}a_{32} + a_{32}^2 \\ 0 & a_{32} & -a_{32} \\ 0 & 0 & 1 \end{pmatrix}.$$

The following result emerges from direct evaluation

$$A_1 = A_0 = \begin{pmatrix} -\tau_1 & -\tau_2 & -\tau_3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where τ_1, τ_2 and τ_3 are the same as in Lemma 4. 1. moreover, Eq.(4.8) can be transformed into the following form

$$J_1 H_1^2 J_1^T + J_1 A J_1^{-1} J_1 \Sigma_1 J_1^T + J_1 \Sigma_1 J_1^T \left(J_1 A J_1^{-1}\right)^T = 0.$$

that is

$$H_0^2 + A_1 \Sigma_0 + \Sigma_0 A_1^T = 0,$$

where $\Sigma_0 = \rho_1^{-2} J_1 \Sigma_1 J_1^T,\ \rho_1 = a_{21}a_{32}\sigma_1.$ It is noticed that the matrix A has all negative real-part eigenvalues, then A_1 is a standard R_1 matrix. According to lemma 4.1, Σ_0 is positive definite whose specific form is as follows :

$$\Sigma_0 = \begin{pmatrix} \dfrac{\tau_2}{2\left(\tau_1\tau_2 - \tau_3\right)} & 0 & -\dfrac{1}{2\left(\tau_1\tau_2 - \tau_3\right)} \\ 0 & 1 & 0 \\ -\dfrac{1}{2\left(\tau_1\tau_2 - \tau_3\right)} & 0 & \dfrac{\tau_1}{2\tau_3\left(\tau_1\tau_2 - \tau_3\right)} \end{pmatrix}.$$

Therefore, $\Sigma_1 = \rho_1^2 J_1 \Sigma_0 \left(J_1^T\right)^{-1}$ is also positive definite.

Step2. Consider the algebraic equation

$$H_2^2 + A\Sigma_2 + \Sigma_2 A^T = 0 \tag{4.9}$$

where $H_2 = diag\left(0, \sigma_2, 0\right)$. Let $A_2 = J_2 A J_2^{-1}$,
where

$$J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

then

$$A_2 = \begin{pmatrix} -a_{21} & a_{21} & a_{21} \\ a_{32} & -a_{32} & 0 \\ 0 & -a_{13} & a_{11} \end{pmatrix}.$$

Through the similarity transformation $A_3 = J_3 A_2 J_3^{-1}$, the standard R_1 matrix properties are preserved, leading directly to the identity $A_3 = A_0$. The corresponding standardization matrix takes the form

$$J_3 = \begin{pmatrix} -a_{13}a_{32} & a_{13}\left(a_{11} + a_{32}\right) & a_{11}^2 \\ 0 & -a_{13} & -a_{11} \\ 0 & 0 & 1 \end{pmatrix}.$$

Eq (4.9) can be equivalently expressed as

$$\begin{aligned} & J_3 J_2 H_2^2 \left(J_3 J_2\right)^T + \left(J_3 J_2\right) A \left(J_3 J_2\right)^{-1} \left(J_3 J_2\right) \Sigma_2 \left(J_3 J_2\right)^T \\ & + \left(J_3 J_2\right) \Sigma_2 \left(J_3 J_2\right)^T \left[\left(J_3 J_2\right) A \left(J_3 J_2\right)^{-1}\right]^T = 0 \end{aligned}.$$

Namely,

$$H_0^2 + A_3 \Sigma_0 + \Sigma_0 A_3^T = 0,$$

where $\Sigma_0 = \rho_2^{-2} \left(J_3 J_2\right) H_2^2 \left(J_3 J_2\right)^T$, $\rho_2 = -a_{13} a_{32} \sigma_2$. Building upon the established positive definiteness of Σ_0, the congruent transformation $\Sigma_2 = \rho_2^2 \left(J_3 J_2\right)^{-1} \Sigma_0 \left[\left(J_3 J_2\right)^T\right]^{-1}$ necessarily inherits this property, confirming Σ_2 is positive definiteness.

Step 3. Consider the algebraic equation

$$H_3^2 + A\Sigma_3 + \Sigma_3 A^T = 0, \tag{4.10}$$

where $H_3 = diag(0,0,\sigma_3)$.

Likewise, let $A_4 = J_4 A J_4^{-1}$, where $J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, thus $A_4 = \begin{pmatrix} -a_{32} & 0 & a_{32} \\ -a_{13} & -a_{11} & 0 \\ a_{21} & a_{21} & -a_{21} \end{pmatrix}$.

Define $A_5 = J_5 J_4 A J_4^{-1} J_5^{-1}$, where $J_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{a_{12}}{a_{13}} & 1 \end{pmatrix}$, we have

$$A_5 = \begin{pmatrix} -a_{32} & \frac{-a_{21}a_{32}}{a_{13}} & a_{32} \\ -a_{13} & -a_{11} & 0 \\ 0 & m & -a_{21} \end{pmatrix},$$

Where $m = a_{21} + \frac{a_{21}^2 - a_{11}a_{21}}{a_{13}}$. Based on the value of m, we will consider the following two cases :

(1) $m \neq 0$; (2) $m = 0$.

Case1: If $m \neq 0$, following the derivation process in step 1, we define $A_6 = J_6 A_5 J_6^{-1}$, where the standard transformation matrix J_6 is given by

$$J_6 = \begin{pmatrix} -ma_{13} & -m(a_{11} + a_{21}) & a_{21}^2 \\ 0 & m & -a_{21} \\ 0 & 0 & 1 \end{pmatrix}.$$

From the canonical properties of the matrix R_1, we deduce the equivalence $A_6 = A_0$. This identity allows us to reformulate Eq.(4.3) as the matrix equation $H_0^2 + A_6\Sigma_0 + \Sigma_0 A_6^T = 0$, where the transformed covariance matrix Σ_0 relates to Σ_3 through the composite operator chain $\Sigma_0 = \rho_3^{-2}(J_6 J_5 J_4)\Sigma_3 (J_6 J_5 J_4)^T$, with the scaling factor $\rho_3 = -a_{13} m \sigma_3$. Consequently, we derive the positive definite structure

$$\Sigma_3 = \rho_3^2 (J_6 J_5 J_4)^{-1} \Sigma_0 \left[(J_6 J_5 J_4)^T \right]^{-1}.$$

Case 2: When the parameter constraint $a_{13} + a_{21} = a_{11}$ holds, i.e. $m = 0$, we establish the transformed operator B_0 through the similarity relation $B_0 = J_7 A_5 J_7^{-1}$, with the canonical transformation operator J_7 having the explicit representation

$$J_7 = \begin{pmatrix} -a_{13} & -a_{11} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By a simple computation, we have

$$B_0 = \begin{pmatrix} -l_1 & -l_2 & -l_2 \\ 1 & 0 & 0 \\ 0 & 0 & -a_{21} \end{pmatrix},$$

where

$$l_1 = a_{11} + a_{32}, l_2 = a_{13}a_{32}. \tag{4.11}$$

By (4.11) and the expression of a_{11}, a_{21} and a_{32}, we can easily validated that $l_1 > 0$ and $l_2 > 0$. That is to say, the condition of Lemma 4. 2 are satisfied, hence A_7 is a standard matrix and the corresponding matrix θ_0 is semi-positive definite. Similarly, we transform (4.10) into the following equation

$$\begin{aligned}&(J_7J_5J_4)H_3^2(J_7J_5J_4)^T + (J_7J_5J_4)A(J_7J_5J_4)^{-1}(J_7J_5J_4)\Sigma_3(J_7J_5J_4)^T \\ &+(J_7J_5J_4)\Sigma_3(J_7J_5J_4)^T\left[(J_7J_5J_4)A(J_7J_5J_4)^{-1}\right]^T = 0\end{aligned}$$

i.e. $H_0^2 + B_0\theta_0 + \theta_0 B_0^T = 0,$ where $\theta_0 = \tilde{\rho}_4^{-2} J_7J_5J_4\Sigma_3(J_7J_5J_4)^T, \tilde{\rho}_4 = -a_{13}\sigma_3.$ From Lemma 4. 2, we have that θ_0 is semi-positive definite, and

$$\theta_0 = \begin{pmatrix} \frac{1}{2l_1} & 0 & 0 \\ 0 & \frac{1}{2l_1l_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The transformed matrix $\Sigma_3 = \tilde{\rho}_4^2(J_7J_5J_4)^{-1}\theta_0\left[(J_7J_5J_4)^{-1}\right]^T$ inherits the positive semidefinite property through this construction. Under the epidemiological threshold condition $R_0^s > 1$, the composite symmetric matrix $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$ appearing in the algebraic system (4.6) becomes strictly positive definite.

The stochastic analysis reveals that the stationary distribution $\pi(\bullet)$ near the endemic equilibrium E^* admits a unique log-normal characterization with probability density function

$$\Phi(s,e,i)=(2\pi)^{-\frac{1}{2}}|\Sigma|^{-\frac{1}{2}}e^{-\frac{1}{2}\left(\ln\frac{s}{s^*},\ln\frac{e}{e^*},\ln\frac{i}{i^*}\right)\Sigma^{-1}\left(\ln\frac{s}{s^*},\ln\frac{e}{e^*},\ln\frac{i}{i^*}\right)^T},$$

where the specific form of the Σ can be determined by the above discussion. This completes the proof. □

5 Extinction

This section outlines key conditions required for disease eradication.

Theorem 5.1 Let $(s(t),e(t),i(t))$ be the solution of the system (1.4) with any initial value $(s(0),e(0),i(0))\in R_+^3$. if $\frac{b+\sigma}{2\delta}\wedge(\mu+q+b)>\frac{1}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)$ and $\tilde{R}_0^s<1$ is satisfied, then the disease will become extinct. where $\tilde{R}_0^s=\frac{2\delta\beta}{(\delta+b)\left(\frac{1}{2}\sigma_2^2\wedge\left(u+q+b+\frac{1}{2}\sigma_3^2\right)\right)}$.

Proof. Prior to demonstrating this theorem, we introduce a twice continuously differentiable Lyapunov function $V:\mathbb{R}_+^3\to\mathbb{R}_+$ defined as $V=e+\frac{\delta+b}{\delta}i$.

Implementing the stochastic chain rule to the composition $\ln V$ results in

$$\begin{aligned}
d\ln V=&\left\{\frac{1}{e+\frac{\delta+b}{\delta}i}\beta is-\frac{1}{\left(e+\frac{\delta+b}{\delta}i\right)^2}\left(\frac{1}{2}\sigma_2^2e^2+\left(\frac{\delta+b}{\delta}\right)^2(u+q+b)i^2\right.\right.\\
&\left.\left.+\frac{\delta+b}{\delta}(u+q+b)ei+\frac{1}{2}\left(\frac{\delta+b}{\delta}\right)^2\sigma_3^2i^2\right)\right\}dt\\
&+\frac{\sigma_2e}{e+\frac{\delta+b}{\delta}i}dB_2(t)+\frac{\frac{\delta+b}{\delta}\sigma_3i}{e+\frac{\delta+b}{\delta}i}dB_3(t)\\
\le&\left\{\frac{\delta\beta}{\delta+b}-\frac{\frac{1}{2}\sigma_2^2\wedge\left(u+q+b+\frac{1}{2}\sigma_3^2\right)}{2}\right\}dt\\
&+\frac{\sigma_2e}{e+\frac{\delta+b}{\delta}i}dB_2(t)+\frac{\frac{\delta+b}{b}\sigma_3i}{e+\frac{\delta+b}{\delta}i}dB_3(t).
\end{aligned}\tag{5.1}$$

The averaged integral transform of Eq.(5.1) over the interval $[0,t]$ shows

$$\begin{aligned}\frac{\ln V(t)}{t} &\le \frac{\ln V(0)}{t} + \frac{\delta\beta}{\delta+b} - \frac{\frac{1}{2}\sigma_2^2 \wedge \left(u+q+b+\frac{1}{2}\sigma_3^2\right)}{2} \\ &+ \frac{\sigma_2}{t}\int_0^t \frac{e(s)}{e(s)+\frac{\delta+b}{\delta}i(s)}dB_2(s) \\ &+ \frac{\frac{\delta+b}{\delta}\sigma_3}{t}\int_0^t \frac{i(s)}{e(s)+\frac{\delta+b}{\delta}i(s)}dB_3(s) \\ &= \frac{\ln V(0)}{t} + \left\{\frac{\delta\beta}{\delta+b} - \frac{\frac{1}{2}\sigma_2^2 \wedge \left(u+q+b+\frac{1}{2}\sigma_3^2\right)}{2}\right\} \\ &+ \frac{M_1(t)}{t} + \frac{M_2(t)}{t},\end{aligned} \tag{5.2}$$

where

$$M_1(t) := \int_0^t \frac{\sigma_2 e(s)}{e(s)+\frac{\delta+b}{\delta}i(s)}dB_2(s),$$

$$M_3(t) := \int_0^t \frac{\frac{\delta+b}{\delta}\sigma_3 i(s)}{e(s)+\frac{\delta+b}{\delta}i(s)}dB_3(s)$$

represent locally martingalic processes with L^2-variation bounds

$$\begin{aligned}\langle M_1, M_1\rangle_t &= \sigma_2^2\int_0^t \left(\frac{e(s)}{e(s)+\frac{\delta+b}{\delta}i(s)}\right)^2 ds, \\ &\le \sigma_2^2 t\end{aligned}$$

$$\begin{aligned}\langle M_2, M_2\rangle_t &= \sigma_3^2\int_0^t \left(\frac{\frac{\delta+b}{\delta}\sigma_3 i(s)}{e(s)+\frac{\delta+b}{\delta}i(s)}\right)^2 ds. \\ &\le \sigma_3^2 t\end{aligned}$$

The strong law for local martingales implies the time-averaged decay

$$\lim_{t\to\infty}\frac{M_j(t)}{t}=0,\quad j=1,2. \tag{5.3}$$

Taking the superior limit on both sides of (5.2) which together with (5.3) leads to that $\limsup_{t\to\infty}\frac{\ln\left(e+\frac{\delta^\alpha+b^\alpha}{\delta^\alpha}i\right)}{t}\le\frac{\frac{1}{2}\sigma_2^2\wedge\left(u+q+b+\frac{\sigma_3^2}{2}\right)}{2}\left(\tilde{R}_0^s-1\right)$, which is desired statement. In addition, if $\tilde{R}_0^s<1$, we can conclude that $\lim_{t\to+\infty}e(t)=0$, $\lim_{t\to+\infty}i(t)=0$.

That is to say, if $\tilde{R}_0^s<1$, $i(t)$ of the disease will tend to extinction. This completes the proof. □

6 Numerical simulation

To computationally validate our analytical results, we implement the stochastic system (1.4) using Milstein's higher-order discretization scheme [24]. The numerical approximation takes the discrete form:

$$\begin{cases} s_{j+1}=\left[b-\beta i_j s_j-bs_j\right]\Delta t+\sigma_1 s_j\sqrt{\Delta t}\varepsilon_{1,j}+\frac{\sigma_1^2}{2}s_j\left(\varepsilon_{1,j}^2-1\right)\Delta t,\\ e_{j+1}=\left[\beta i_j s_j-(\delta+b)e_j\right]\Delta t+\sigma_2 e_j\sqrt{\Delta t}\varepsilon_{2,j}+\frac{\sigma_1^2}{2}e_j\left(\varepsilon_{2,j}^2-1\right)\Delta t,\\ i_{j+1}=\left[\delta e_j-(u+q+b)i_j\right]\Delta t+\sigma_3 i_j\sqrt{\Delta t}\varepsilon_{3,j}+\frac{\sigma_3^2}{2}i_j\left(\varepsilon_{3,j}^2-1\right)\Delta t. \end{cases} \tag{6.1}$$

the computational framework utilizes fixed time steps $\Delta t>0$, with $\varepsilon_{k,j}\,(k=1,2,3)$ representing independent standard normal random variates across all discrete points $j=1,\cdots,n.$

Example6.1. To computationally establish the existence of an ergodic stationary distribution, we initialize the system with $s(0)=0.8, e(0)=0, i(0)=0.2, r(0)=0$, the parameter configuration is specified as $b=2\times10^{-4}$, $\beta=10, q=0.861469$, $\sigma_1=\sigma_2=\sigma_3=0.01$, the stochastic reproduction number evaluates to

$$R_0^s:=\frac{b\delta\beta}{\left(b+\frac{1}{2}\sigma_1^2\right)\left(\delta+b+\frac{1}{2}\sigma_2^2\right)\left(u+q+b+\frac{1}{2}\sigma_3^2\right)}=1.5596>1.$$

The established conditions validate the requirements of Theorem 3. 1, guaranteeing existence of a unique stationary distribution $\pi(\bullet)$ for system (1.4). This ergodic property ensures pathogen persistence with probability one, supported by numerical evidence in Figure 1.

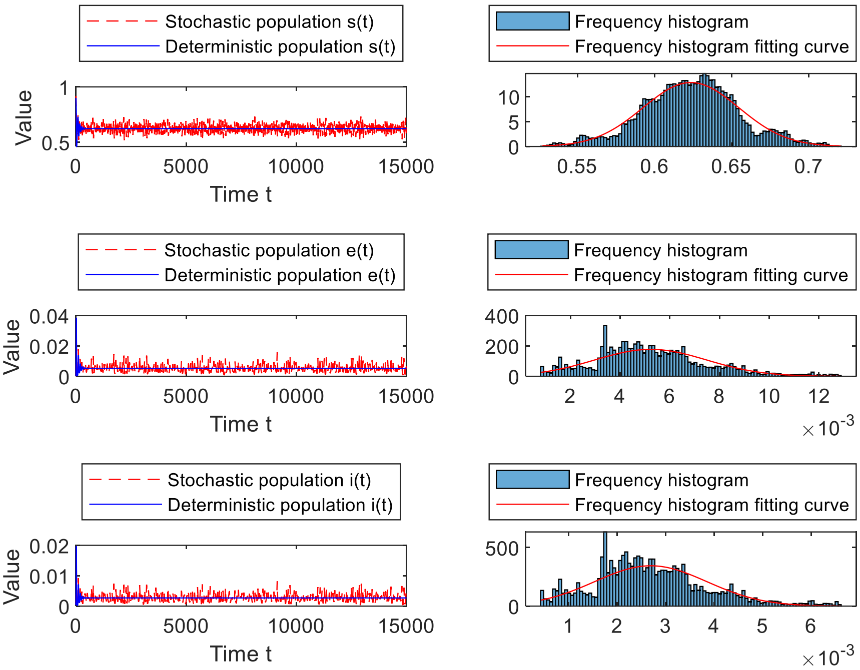

Figure 1: The left column shows the time series diagrams of s(t),e(t) and i(t) in the stochastic model (1.3) and their corresponding deterministic model (1.1). The right column displays the frequency histogram and each fitting curve.

The time series plots provide a clear visualization of the temporal changes in susceptible, exposed, and infected rate of populations during disease spread. The stochastic model incorporates environmental noise and other random factors, offering a more realistic representation of real-world transmission dynamics. In contrast, the deterministic model assumes fixed parameters, providing an idealized scenario. The comparison between the two models highlights the potential impact of stochasticity on disease transmission, suggesting that random factors can introduce variability and unpredictability in the spread of the disease.

This finding has critical implications for public health, as it suggests that natural transmission dynamics alone are insufficient to eradicate the disease. Continuous intervention measures, such as vaccination, quarantine, or treatment, are necessary to control and mitigate the spread.

In summary, the results underscore the importance of incorporating stochastic factors into disease models to better inform public health policies and control measures.

Example 6.2. Let $s(0)=0.8, e(0)=0, i(0)=0.1, r(0)=0, b=2\times 10^{-3}$, $\beta=0.2, q=0.4$, $\delta=0.8$, $\mu=0.6$, $\sigma_1=0.1, \sigma_2=1, \sigma_3=1$, then by direct calculation, we can get then by direct calculation, we can get

$$R_0^s=\frac{2(\delta+b)\delta\beta}{\frac{1}{2}\sigma_2^2\wedge(\delta+b)^2\left(\mu+q+b+\frac{1}{2}\sigma_3^2\right)}=0.802<1.$$

In other words, the condition of Theorem 5.1 is satisfied. Obviously, both the infected and exposed classes tend to zero, which means the disease will go to extinction exponentially with probability one. Fig.2. shows this.

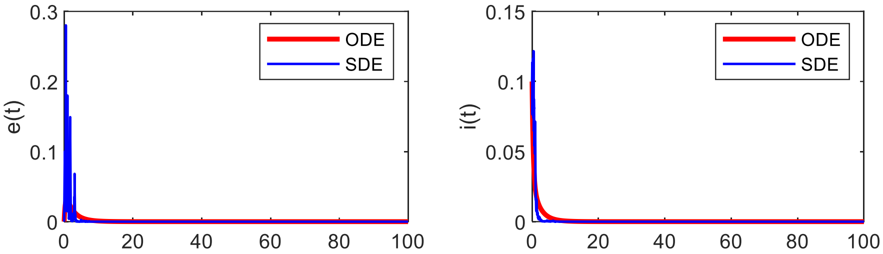

Figure 2: Time-series trajectories illustrating disease extinction in the stochastic SEIR model.

This implies that, under the given conditions, the disease cannot sustain itself in the population and will eventually die out.These fluctuations account for environmental variability and other random effects that can influence disease transmission. The figure demonstrates that even with these stochastic factors, the disease tends to extinction when $R_0^s < 1$, highlighting the robustness of this threshold condition.The findings underscore the importance of achieving a basic reproduction number below one to ensure disease eradication. This can be achieved through interventions such as vaccination, quarantine, and treatment, which reduce transmission rates (β) or increase recovery rates (μ). The figure 2 provides a visual confirmation that such measures can effectively drive the disease toward extinction.

In summary, the figure and the associated analysis provide a clear illustration of the conditions under which a disease will decline and eventually disappear from a population. The results emphasize the critical role of stochastic modeling in understanding disease dynamics and the importance of maintaining $R_0^s < 1$ for effective disease control and eradication.

7 Conclusion

This work presents a stochastic SEIR epidemic model incorporating therapeutic intervention. Our analytical investigation establishes several fundamental results: (i) Existence and Uniqueness: For arbitrary positive initial conditions, the stochastic system (1.4) admits a unique globally positive solution that persists for all time. (ii) Ergodic Properties: Through construction of appropriate Lyapunov functionals, we derive verifiable sufficient conditions guaranteeing the existence of a unique stationary distribution $\pi(\bullet)$ possessing ergodic characteristics. Notably, these conditions simultaneously enable the explicit characterization of the probability density function governing fluctuations about the quasi-endemic equilibrium state. (iii) Extinction Thresholds: We establish quantitatively precise criteria that determine disease eradication, providing a stochastic counterpart to the basic reproduction number in deterministic epidemic models.

Numerical simulations under biologically relevant parameters reveal two key dynamical regimes: (i) Stochastic Perturbation Analysis: Moderate environmental variability induces bounded oscillations about endemic equilibrium, demonstrating asymptotic stability in distribution with minimal population fluctuations. (ii) Noise-Induced Extinction Phenomenon: Sufficiently large diffusion coefficients σ_i drive pathogen elimination, establishing a stochastic bifurcation threshold where environmental noise dominates epidemic persistence.

Several important research directions warrant further investigation. First, while the local probability density function for system (1.4) has been analytically characterized, the derivation of its global counterpart remains an open problem that requires rigorous mathematical treatment. This fundamental question represents a significant challenge in stochastic dynamical systems theory. Our forthcoming investigation will prioritize extending the stochastic epidemic system (1.4) into singular kernel operator frameworks, specifically analyzing its non-Markovian dynamics through Caputo fractional differential operators.

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Declaration of interest statement

The author reports there are no competing interests to declare.

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